# FINAL: ALGEBRAIC GEOMETRY

Date: $21^{\text{st}}$ **April 2025**

The Total points is **110** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (10 points) Let $k$ be a field. Let $f : A \to B$ be a $k$-algebra homomorphism between finitely generated $k$-algebras $A$ and $B$. Show that $f^{-1}(m)$ is a maximal ideal of $A$ for every maximal ideal $m$ of $B$.

(2) (5+15=20 points) Let $X$ be an affine variety over an algebraically closed field $k$. Define the function field of $X$. Show that there exists $n$ such that the function field of $X$ is a finite field extension of the field $k(x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are algebraically independent.

(3) (5+10=15 points) Let $X$ be an algebraic subset of a projective space $\mathbb{P}^n$. When is $X$ called irreducible? Assuming $X$ to be an algebraic subset of a projective space, show that $X$ is irreducible iff the homogeneous coordinate ring of $X \subset \mathbb{P}^n$ is a domain.

(4) (15 points) Which of the following expression define a morphism between projective varieties? Give reasons.
  (a) $f : \mathbb{P}^2 \to \mathbb{P}^3$ where $f([a, b, c]) = [a + b^2, ab + c^2, abc, b + ca]$.
  (b) $f : \mathbb{P}^2 \to \mathbb{P}^3$ where $f([a, b, c]) = [a^2b + abc, bc^2 - b^2c, a^3 - b^3 + c^3, a^3]$.
  (c) $f : \mathbb{P}^2 \to \mathbb{P}^3$ where $f([a, b, c]) = [a + b, b + c, c + a, a + b + c]$.

(5) (5+10+10=25 points)Let $f = x^7 - y^7 + z^7$ be an irreducible homogeneous polynomial in $\mathbb{C}[x, y, z]$ and let $X = Z(f)$ be the projective curve defined by $f$.
  (a) Find the locus where the rational function $\bar{x}/\bar{y}$ on $X$ is regular.
  (b) Find lines $L_1$ and $L_2$ in $\mathbb{P}^2$ such that $X \cap L_1$ contains exactly one point and $X \cap L_2$ contains exactly six points.
  (c) Find a nonconstant rational function on $X$ which is regular away from the point $[1, 1, 0]$.

(6) (4+4+10+7=25 points) Let $X$ be a smooth projective curve over $\mathbb{C}$.
  (a) What is an effective divisor on $X$?
  (b) Let $D$ be an effective divisor on $X$. Define $L(D)$.
  (c) Show that there exist an effective divisor $D_1$ on $X$ linearly equivalent to $D$ but different from $D$ iff $L(D)$ has dimension at least 2.
  (d) When $X = Z(x^7 - y^7 + z^7)$, find an effective divisor $D$ on $X$ such that $L(D)$ has dimension at least 2.